and the equation of state

$$I = I_0 \Big\{1 + \sum_{q=1}^{\infty} \Big[(-1)^q (q!)^{-1} \Big[\prod_{n=1}^{q} (\gamma + n - 2)\Big] \Big(-1 + \sum_{m=0}^{\infty} t^m \sum_{l=0}^{m} \sum_{k=0}^{l} \delta_{ijs} x_{i,\bar{1}}^{k} x_{j,\bar{2}}^{l-k} x_{k,\bar{3}}^{m-l}\Big)^q\Big]\Big\}$$

From these recurrent relationships we can obtain (by equating terms with equal powers of t) the expressions for coefficients of series (3.7). The analysis of the latter shows their convergence, since they are majorized by the convergent series

$$|M| [1 / (1\cdot 2) + 1 / (2\cdot 3) + \ldots]$$

The radius of convergence of series (3.7) is determined by the interval $0 < t < |M^{-1}|$, where M is the maximum value of the m-th derivative of input data. The existence of solution of the Cauchy problem is thereby proved. Its uniqueness follows from the uniqueness of the specification of input data.

Thus Weber equations in some particular cases (stationary periodic flows) yield very simple equations which can be solved by conventional analytical methods.

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THE METHOD OF SUCCESSIVE APPROXIMATIONS IN PROBLEMS OF THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER

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N. N. SHAKHOV and IU. D. SHEVELEV
(Moscow)

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The analytical method of calculation of a three-dimensional boundary layer in a compressible fluid stream is considered. The method is based on the use of successive approximations and is similar to that used in the case of incompressible

fluid [1]. Results of this approximate analysis are compared with those obtained by the method of finite differences.

1. In an arbitrary system of curvilinear coordinates attached to a streamlined body the system of equations defining the three-dimensional boundary layer in a compressible fluid is of the form [2]

$$Lu + A_1u^2 + A_2\omega^2 + A_3u\omega = A_4 + \frac{1}{\rho}\frac{\partial}{\partial\zeta}\left(\mu\frac{\partial u}{\partial\zeta}\right), \quad \frac{\partial p}{\partial\zeta} = 0 \qquad (1.1)$$

$$L\omega + B_1u^2 + B_2\omega^2 + B_3u\omega = B_4 + \frac{1}{\rho}\frac{\partial}{\partial\zeta}\left(\mu\frac{\partial\omega}{\partial\zeta}\right)$$

$$\frac{\partial}{\partial\zeta}\left(\rho\sqrt{\frac{g}{g_{11}}}u\right) + \frac{\partial}{\partial\eta}\left(\rho\sqrt{\frac{g}{g_{22}}}\omega\right) + \sqrt{g}\frac{\partial\rho v}{\partial\zeta} = 0$$

$$\rho LH = \frac{\partial}{\partial\zeta}\left\{\frac{\mu}{\sigma}\left[\frac{\partial H}{\partial\zeta} + (\sigma - 1)\frac{\partial}{\partial\zeta}\frac{U^2}{2}\right]\right\}$$

$$L \equiv \frac{u}{\sqrt{g_{11}}}\frac{\partial}{\partial\xi} + \frac{\omega}{\sqrt{g_{22}}}\frac{\partial}{\partial\eta} + v\frac{\partial}{\partial\zeta}, \quad g = g_{11}g_{22} - g_{12}^2$$

where g_{ij} are components of the metric tensor; u, ω and v are velocity components along the axes ξ, η and ζ ; H is the total enthalpy; ρ is the density; T is the temperature; p is the pressure; μ is the viscosity coefficient; c_p is the specific heat at constant pressure, and σ is the Prandtl number. The system of equations (1.1) is closed by the equation of state $p = \rho RT$. Coefficients A_i and B_i $(i = 1,2,3,4)$ are determined by the geometry of the body and the external flow [2].

System (1.1) is solved for the following boundary conditions:

$$u = \omega = v = 0, \qquad H = H_0 \quad \text{for} \quad \zeta = 0 \qquad (1.2)$$

$$u \to u_e, \ \omega \to \omega_e, \qquad H \to H_e \quad \text{for} \quad \zeta \to \infty$$

where the subscripts e and 0 relate to parameters of the stream at the external boundary of the boundary layer and at the surface of the body. These boundary conditions correspond to a boundary layer of an isentropic external flow.

Using transformations similar to those of Dorodnitsyn we reduce system (1.1) to the form convenient for integration

$$\xi_1 = \xi, \quad \eta_1 = \eta, \quad \lambda = \sqrt{\frac{u_e}{\mu_e\rho_e\alpha}}\int_0^\zeta \rho\, d\zeta \qquad (1.3)$$

where $\alpha(\xi, \eta)$ is a certain function whose selection will be discussed below. We pass from the unknown functions u, ω, v and H to the new variables $E(\xi_1, \eta_1, \lambda)$, $G(\xi_1, \eta_1, \lambda)$, $K(\xi_1, \eta_1, \lambda)$ and $\theta(\xi_1, \eta_1, \lambda)$ by using formulas

$$u = u_e(\xi, \eta)\, E(\xi, \eta, \lambda), \ \omega = \beta(\xi, \eta)\, u_e(\xi, \eta)\,(G + \varphi E) \qquad (1.4)$$

$$\rho v + \frac{u}{\sqrt{g_{11}}}\frac{\partial\zeta_1}{\partial\xi} + \frac{\omega}{\sqrt{g_{22}}}\frac{\partial\zeta_1}{\partial\eta} = \sqrt{\mu_e\rho_e u_e}\left(K - \frac{\alpha}{\sqrt{g_{11}}}E\frac{\partial\lambda}{\partial\xi} - \right.$$

$$\left. \frac{\alpha\beta}{\sqrt{g_{22}}}(G + \varphi E)\frac{\partial\lambda}{\partial\eta}\right), \quad H = H_0 + (H_e - H_0)\theta$$

$$\left(\zeta_1 = \int_0^\zeta \rho\, d\zeta, \quad \varphi = \frac{\omega_e}{\beta u_e}\right)$$

where $\beta(\xi, \eta)$ is an arbitrary function of variables ξ and η. Here and subsequently subscripts 1 at ξ_1 and η_1 are omitted.

As the result of substitutions (1.3) and (1.4) system (1.1) assumes the form

$$\frac{\partial}{\partial\lambda}\left(l\frac{\partial E}{\partial\lambda}\right) = K\frac{\partial E}{\partial\lambda} + N_1^*(E^2 - F) + N_2^*G^2 + N_3^*EG + N_4E\frac{\partial E}{\partial\xi} + N_5(G + \varphi E)\frac{\partial E}{\partial\eta} \quad (1.5)$$

$$\frac{\partial}{\partial\lambda}\left(l\frac{\partial G}{\partial\lambda}\right) = K\frac{\partial G}{\partial\lambda} + M_1^*(E^2 - F) + M_2^*G^2 + M_3^*EG + N_4E\frac{\partial G}{\partial\xi} + N_5(G + \varphi E)\frac{\partial G}{\partial\eta}$$

$$\frac{\partial}{\partial\lambda}\left(\frac{l}{\sigma}\frac{\partial\theta}{\partial\lambda}\right) = K\frac{\partial\theta}{\partial\lambda} + \frac{\partial}{\partial\lambda}\left\{\frac{1-\sigma}{\sigma(1-t_0)}\frac{l}{k}\frac{\partial}{\partial\lambda}(E^2 + \beta^2(G+\varphi E)^2 + 2\beta\cos\psi_0(G+\varphi E)E)\right\} + N_4(1-\theta)E\frac{1}{1-t_0}\frac{\partial t_0}{\partial\xi} + N_5(1-\theta)\times (G+\varphi E)\frac{1}{1-t_0}\frac{\partial t_0}{\partial\eta} + N_4E\frac{\partial\theta}{\partial\xi} + N_5(G+\varphi E)\frac{\partial\theta}{\partial\eta}$$

$$-\frac{\partial K}{\partial\lambda} = P_1^*E + P_2^*G + N_4\frac{\partial E}{\partial\xi} + N_5\frac{\partial G}{\partial\eta} + \varphi N_5\frac{\partial E}{\partial\eta}$$

where

$$t_0 = \frac{H_0}{H_e}, \quad F = \frac{\rho_e}{\rho} = 1 + (1-t_0)(\theta - 1)\left(1 + \frac{\gamma-1}{2}M_e^2\right) - \frac{\gamma-1}{2}M_e^2\left(\frac{E^2 + \beta^2(G+\varphi E)^2 + 2\beta\cos\psi_0 E(G+\varphi E)}{1 + \beta^2\varphi^2 + 2\beta\varphi\cos\psi_0} - 1\right)$$

$$l = \frac{\mu\rho}{\mu_e\rho_e}$$

$$k = (1 + 2\beta\varphi\cos\psi_0 + \beta^2\varphi^2)(1 + 2/(\gamma-1)M_e^2)$$

$$\cos\psi_0 = g_{12}/\sqrt{g_{11}g_{22}}$$

where γ is the ratio of specific heats ($\gamma = c_p/c_v$), M_e is the Mach number at the external boundary of the boundary layer, and σ is the Prandtl number. Coefficients N_1^*, N_2^*, N_3^*, M_1^*, M_2^*, M_3^*, P_1^* and P_2^* depend only on ξ and η. The expressions for these coefficients appear in [2].

As the result of transformations the boundary conditions assume the form

$$E = G = K = \theta = 0 \quad \text{for} \quad \lambda = 0 \quad (1.6)$$

$$E \to 1, \quad G \to 0, \quad \theta \to 1 \quad \text{for} \quad \lambda \to \infty \quad (1.7)$$

The components of friction at the wall and the heat flux are determined by formulas ($g_{12} = 0$)

$$\tau_1 = \mu\frac{\partial u}{\partial\zeta}\bigg|_{\zeta=0}, \qquad \tau_2 = \mu\frac{\partial\omega}{\partial\zeta}\bigg|_{\zeta=0}, \qquad q = \lambda\frac{\partial T}{\partial\zeta}\bigg|_{\zeta=0}$$

Integrating the equations of system (1.5) with respect to the λ-coordinate from some value of that coordinate to infinity and allowing for the boundary conditions (1.7), we obtain

$$-l\frac{\partial E}{\partial \lambda} = -K(E-1) + (P_1^* + N_1^*)\theta_{11} + N_1^*\theta_1 + \tag{1.8}$$

$$(P_2^* + N_3^*)\theta_{21} + N_1^*\int_\lambda^\infty (1-F)\,d\lambda + N_2^*\theta_{22} - P_2^*\theta_2 +$$

$$N_4\frac{\partial\theta_{11}}{\partial\xi} + N_5\frac{\partial\theta_{12}}{\partial\eta} + \varphi N_5\frac{\partial\theta_{11}}{\partial\eta}$$

$$-l\frac{\partial G}{\partial\lambda} = -KG + M_1^*\left(\theta_{11} + \theta_1 + \int_\lambda^\infty (1-F)\,d\lambda\right) + (P_2^* +$$

$$M_2^*)\theta_{22} + (P_1^* + M_3^*)\theta_{21} + N_4\frac{\partial\theta_{21}}{\partial\xi} + N_5\frac{\partial\theta_{22}}{\partial\eta} + \varphi N_5\frac{\partial\theta_{21}}{\partial\eta}$$

$$-\frac{l}{\sigma}\frac{\partial\theta}{\partial\lambda} = -K(\theta-1) + \frac{1-\sigma}{\sigma}\frac{l}{k(1-t_0)}\frac{\partial}{\partial\lambda}[E^2 + \beta^2(G+\varphi E)^2 +$$

$$2\beta\cos\psi_0(G+\varphi E)E] + P_1^*\theta_{31} + P_2^*\theta_{32} +$$

$$N_4\frac{\partial}{\partial\xi}\theta_{31} + N_5\frac{\partial\theta_{32}}{\partial\eta} + \varphi N_5\frac{\partial\theta_{31}}{\partial\eta} + N_4\theta_{31}\frac{\partial\ln(1-t_0)}{\partial\xi} +$$

$$N_5(\theta_{32} + \varphi\theta_{31})\frac{\partial\ln(1-t_0)}{\partial\eta}$$

$$-\frac{\partial K}{\partial\lambda} = P_1^*E + P_2^*G + N_4\frac{\partial E}{\partial\xi} + N_5\frac{\partial G}{\partial\eta} + \varphi N_5\frac{\partial E}{\partial\eta}$$

where

$$\int_\lambda^\infty (1-F)\,d\lambda = p_1(\theta_1 + \theta_{11}) + \delta_1\theta_{21} + \delta_2\theta_{22} - (1-t_0)(1+p_1)\theta_3$$

$$p_1 = \frac{U_e^2}{2H_e - U_e^2} = (\gamma-1)\frac{M_e^2}{2}, \qquad \delta_1 = \frac{2\beta p_1(\cos\psi_0 + \beta\varphi)}{(1+\beta^2\varphi^2 + 2\beta\varphi\cos\psi_0)}$$

$$\delta_2 = \beta^2 p_1 / (1 + \beta^2\varphi^2 + 2\beta\varphi\cos\psi_0)$$

Parameters θ_1, θ_2, θ_3, θ_{11}, θ_{12}, θ_{21}, θ_{22}, θ_{31} and θ_{32} are defined by

$$\theta_1 = \int_\lambda^\infty (E-1)\,d\lambda, \quad \theta_2 = \int_\lambda^\infty G\,d\lambda, \quad \theta_3 = \int_\lambda^\infty (\theta-1)\,d\lambda$$

$$\theta_{11} = \int_\lambda^\infty (E-1)E\,d\lambda, \quad \theta_{12} = \int_\lambda^\infty (E-1)G\,d\lambda, \quad \theta_{21} = \int_\lambda^\infty EG\,d\lambda$$

$$\theta_{22} = \int_\lambda^\infty G^2\,d\lambda, \quad \theta_{31} = \int_\lambda^\infty (\theta-1)E\,d\lambda, \quad \theta_{32} = \int_\lambda^\infty (\theta-1)G\,d\lambda$$

We integrate system (1.8) with respect to the λ-coordinate from zero to a certain value of λ, taking into account the boundary condition (1.6) and set $\sigma = \text{const}$. We then have the following expressions for velocity components and the enthalpy profile:

$$-E=\theta_{10}^*+(P_1^*+N_1^*)\theta_{11}^*+N_1^*\theta_1^*+N_2^*\theta_{22}^*+ \tag{1.9}$$
$$(P_2^*+N_3^*)\theta_{21}^*-P_2^*\theta_2^*+N_1^*\theta_{01}^*+N_4\frac{\partial\theta_{11}^*}{\partial\xi}+N_5\frac{\partial\theta_{12}^*}{\partial\eta}+$$
$$\varphi N_5\frac{\partial\theta_{11}^*}{\partial\eta}-N_4\left\langle\theta_{11}\frac{\partial l^{-1}}{\partial\xi}\right\rangle-N_5\left\langle\theta_{12}\frac{\partial l^{-1}}{\partial\eta}\right\rangle-\varphi N_5\left\langle\theta_{11}\frac{\partial l^{-1}}{\partial\eta}\right\rangle$$
$$-G=\theta_{20}^*+M_1^*(\theta_{11}^*+\theta_1^*)+(P_2^*+M_2^*)\theta_{22}^*+$$
$$(P_1^*+M_3^*)\theta_{21}^*+M_1^*\theta_{01}^*+N_4\frac{\partial\theta_{21}^*}{\partial\xi}+N_5\frac{\partial\theta_{22}^*}{\partial\eta}+\varphi N_5\frac{\partial\theta_{21}^*}{\partial\eta}-$$
$$N_4\left\langle\theta_{21}\frac{\partial l^{-1}}{\partial\xi}\right\rangle-N_5\left\langle(\theta_{22}+\varphi\theta_{21})\frac{\partial l^{-1}}{\partial\eta}\right\rangle$$
$$-\frac{\theta}{\sigma}=\theta_{30}^*+\frac{1-\sigma}{\sigma}\frac{1}{k(1-t_0)}(E^2+\beta^2(G+\varphi E)^2+2\beta\cos\psi_0 E\times$$
$$(G+\varphi E))+P_1^*\theta_{31}^*+P_2^*\theta_{32}^*+N_4\frac{\partial\theta_{31}^*}{\partial\xi}+N_5\frac{\partial\theta_{32}^*}{\partial\eta}+\varphi N_5\frac{\partial\theta_{31}^*}{\partial\eta}+$$
$$N_4\frac{\partial\ln(1-t_0)}{\partial\xi}\theta_{31}^*+N_5\frac{\partial\ln(1-t_0)}{\partial\eta}(\theta_{32}^*+\varphi\theta_{31}^*)-$$
$$N_4\left\langle\theta_{31}\frac{\partial l^{-1}}{\partial\xi}\right\rangle-N_5\left\langle(\theta_{32}+\varphi\theta_{31})\frac{\partial l^{-1}}{\partial\eta}\right\rangle$$
$$-K=P_1^*\langle E\rangle+P_2^*\langle G\rangle+N_4\frac{\partial}{\partial\xi}\langle E\rangle+N_5\frac{\partial}{\partial\eta}\langle G\rangle+\varphi N_5\frac{\partial}{\partial\eta}\langle E\rangle$$
$$\theta_{10}^*=-\langle l^{-1}K(E-1)\rangle,\quad \theta_{20}^*=-\langle l^{-1}KG\rangle,\quad \theta_0^*=-\langle l^{-1}K(\theta-1)\rangle$$
$$\theta_1^*-\langle l^{-1}\theta_1\rangle,\quad \theta_2^*=\langle l^{-1}\theta_2\rangle,\quad \theta_3^*=\langle l^{-1}\theta_3\rangle,\quad \theta_{11}^*=\langle l^{-1}\theta_{11}\rangle$$
$$\theta_{12}^*=\langle l^{-1}\theta_{12}\rangle,\quad \theta_{21}^*=\langle l^{-1}\theta_{21}\rangle,\quad \theta_{22}^*=\langle l^{-1}\theta_{22}\rangle,\quad \theta_{31}^*=\langle l^{-1}\theta_{31}\rangle$$

where

$$\langle f_1\rangle=\int_0^\lambda f_1 d\lambda,\quad \theta_{01}^*=p_1(\theta_1^*+\theta_{11}^*)+\delta_1\theta_{21}^*+\delta_2\theta_{22}^*-(1-t_0)(1+p_1)\theta_3^*$$

Solution of the derived nonlinear system of integro-differential equations with boundary conditions

$$E\to 1,\quad \theta\to 1,\quad G\to 0\quad \text{for}\quad \lambda\to\infty \tag{1.10}$$

is equivalent to the solution of system (1.5) with boundary conditions (1.6) and (1.7).

System (1.9) with boundary conditions (1.10) is a complex nonlinear system of integro-differential equations, which we solve by the method of successive approximations, as in the case of an incompressible fluid [1].

2. Let us consider the system of equations (1.9). We neglect in that system the terms which contain derivatives with respect to coordinates ξ and η. Self-similar problems in a specific class of external flows reduce to this case. Hence the proposed method (let us call it the localized self-similar approximation) is valid in the case of self-similar solutions. The nature of solution variations is determined by coefficients M^*_i, N^*_i and P^*_k, which appear in the system of equations (2.2).

The use of the method od successive approximations for solving boundary value problems presents certain difficulties. Let us assume that the $(n-1)$-th approximation

$$E^{(n-1)}(\xi,\eta,\lambda),\qquad G^{(n-1)}(\xi,\eta,\lambda),\qquad \theta^{(n-1)}(\xi,\eta,\lambda)$$

is known.

We substitute these functions into the right-hand parts of system (1.9), and after appropriate integration obtain the n-th approximation. The boundary conditions of the problem must be satisfied at each step of the iteration process. To have the boundary conditions (1.10) satisfied by the n-th approximation we introduce into the $(n-1)$-th approximation the unknown controlling functions in such a way as to have the boundary conditions approximately satisfied by that approximation.

We substitute

$$E^{(n-1)}(\xi, \eta, c\lambda),\ bG^{(n-1)}(\xi, \eta, c\lambda),\ d[\theta^{(n-1)}(\xi, \eta, c\lambda) - E(n-1)(\xi, \eta, \lambda)]$$
$$(c = c(\xi, \eta),\ b = b(\xi, \eta),\ d = d(\xi, \eta))$$

in the right-hand parts of equations of system (1.9), respectively, for

$$E^{(n-1)}(\xi, \eta, \lambda),\ G^{(n-1)}(\xi, \eta, \lambda),\ \theta^{(n-1)}(\xi, \eta, \lambda) - E^{(n-1)}(\xi, \eta, \lambda)$$

Such introduction of controlling functions results in the boundary for E, G, and $E-\theta$ being maintained.

In the localized self-similar approximation the system of equations (1.9) assumes the form

$$-E_a^{(n+1)} = \delta^{(n)}(A_{1a}^{(n)} + b^{(n)}B_{1a}^{(n)} + b^{(n)2}C_{1a}^{(n)} + d^{(n)}D_{1a}^{(n)}) \qquad (2.1)$$
$$-G_a^{(n+1)} = \delta^{(n)}(A_{2a}^{(n)} + b^{(n)}B_{2a}^{(n)} + b^{(n)2}C_{2a}^{(n)} + d^{(n)}D_{2a}^{(n)})$$
$$\frac{\theta_a^{(n+1)}}{\sigma} + \frac{1-\sigma}{k\sigma(1-t_0)}(E^2 + \beta^2(G+\varphi\bar{E})^2 + 2\beta\cos\psi_0)\times$$
$$(G+\varphi E)E_a^{(n+1)} = \delta^{(n)}(A_{3a}^{(n)} + b^{(n)}B_{3a}^{(n)} + d^{(n)}C_{3a}^{(n)} + b^{(n)}d^{(n)}D_{3a}^{(n)})$$

First approximation expressions for coefficients $A_{1a}^{(n)}$, $B_{1a}^{(n)}$, $C_{1a}^{(n)}$, $D_{1a}^{(n)}$, $A_{2a}^{(n)}$, etc. are given below.

For the determination of the unknown controlling functions (c, b, d) we obtain the system of algebraic equations

$$\delta^{(n)} = (-1)(A_{1a\infty}^{(n)} + b^{(n)}B_{1a\infty}^{(n)} + b^{(n)2}C_{1a\infty}^{(n)} + d^{(n)}D_{1a\infty}^{(n)})^{-1} \qquad (2.2)$$
$$b^{(n)2}C_{2a\infty}^{(n)} + b^{(n)}B_{2a\infty}^{(n)} + A_{2a\infty}^{(n)} + d^{(n)}D_{2a\infty}^{(n)} = 0$$
$$\frac{\sigma^*}{\delta^{(n)}} = A_{3a\infty}^{(n)} + b^{(n)}B_{3a\infty}^{(n)} + d^{(n)}C_{3a\infty}^{(n)} + b^{(n)}d^{(n)}D_{3a\infty}^{(n)}$$
$$(\sigma^* = 1/\sigma + (1-\sigma)/[(1-t_0)\sigma(1+2/(\gamma-1)M_e^2)])$$

and for the parameters which define the thermal flux to the body and friction at the wall we have the expressions

$$-l_0\frac{\partial E^{(n+1)}}{\partial\lambda}\bigg|_{\lambda=0} = \sqrt{\delta^{(n)}}\{[N_1^*(1+p_1)(\theta_{11.0}^{(n)} + \theta_{1.0}^{(n)} - (1-t_0)\theta_{3.0}^{(n)}) + \qquad (2.3)$$
$$P_1^*\theta_{11.0}^{(n)}] + b^{(n)}[(P_2^* + N_3^* + \delta_1 N_1^*)\theta_{21.0}^{(n)} - P_2^*\theta_{2.0}^{(n)}] +$$
$$b^{(n)2}(N_2^* + \delta_2 N_1^*)\theta_{22}^*\}$$
$$-l_0\frac{\partial G^{(n+1)}}{\partial\lambda}\bigg|_{\lambda=0} = \sqrt{\delta^{(n)}}\{(1+p_1)(\theta_{11.0}^{(n)} + \theta_{1.0}^{(n)} - (1-t_0)\theta_{3.0}^{(n)})M_1^* +$$
$$b^{(n)}(P_1^* + M_3^* + \delta M_1^*)\theta_{21.0}^{(n)} + b^{(n)2}(P_2^* + M_2^* + \delta_2 M_1^*)\theta_{22.0}^{(n)}\}$$

$$-l_0 \frac{\partial\theta^{(n+1)}}{\partial\lambda}\Big|_{\lambda=0} = \sqrt{\delta^{(n)}}\{(P_1{}^* - R_1{}^*) \times$$
$$(\theta_{11.0}^{(n)} + d^{(n)}\theta_{13.0}^{(n)}) + b^{(n)}(P_2{}^* - R_2{}^*)(\theta_{12.0}^{(n)} + d^{(n)}\theta_{23.0}^{(n)})]$$
$$R_1{}^* = \frac{\alpha}{\sqrt{g_{11}}}\frac{1}{(1-t_0)}\frac{\partial t_0}{\partial\xi} + \frac{\alpha\beta}{\sqrt{g_{22}}}\varphi\frac{1}{(1-t_0)}\frac{\partial t_0}{\partial\eta}$$
$$R_2{}^* = \frac{\alpha\beta}{\sqrt{g_{22}}}\frac{1}{(1-t_0)}\frac{\partial t_0}{\partial\eta}$$

Let us assume that the n-th approximation is known. Then the $(n+1)$-th approximation in the localized self-similar case is determined by the above formulas. We determine $\delta^{(n)}$, $b^{(n)}$ and $d^{(n)}$ by the algebraic system and then find the velocity and enthalpy profiles, the resistance due to friction at the wall, and the heat flux to the wall. Note that for $M_e \equiv 0$ and $t_0 \equiv 1$, the formulas which define friction are of the same form as in the case of incompressible fluid.

We eliminate $\delta^{(n)}$ from the first and third of Eqs. (2.2) and determine the quantity $d^{(n)}$. The substitution of $d^{(n)}$ into the second of Eqs. (2.2) shows that the quantity $b^{(n)}$ is determined by the solution of a cubic equation. Various cases can be obtained by solving the cubic equation for $b^{(n)}$ and this results in the indeterminacy of solution, as in the case of incompressible fluid.

Coefficients A_1, A_2, A_3, etc. are double or iterated integrals of functions E, G and θ. The convergence rate of the iteration process depends on the selection of the initial approximation, which in the solution of complicated problems predetermines the success of the process, since it is necessary to obtain a reasonably accurate solution even in the first approximation.

3. Let us specify the zero approximation as follows:

$$E^{(0)} = 1 - z_0(\zeta), \qquad G^{(0)} = b^{(0)}(z_0(\zeta) - z_{-1}(\zeta))$$
$$\theta^{(0)} = 1 - z_0(\zeta) + d^{(0)}(z_0(\zeta) - z_{-1}(\zeta))$$

where

$$\zeta = c(\xi, \eta)\lambda = \lambda/\sqrt{\delta^{(0)}}$$

On the basis of (2.1) the solution in the localized self-similar approximation $E^{(1)}$, $G^{(1)}$ and $\theta^{(1)}$ is defined by

$$-E_a^{(1)} = \delta^{(0)}(A_{1a}^{(0)} + b^{(0)}B_{1a}^{(0)} + b^{(0)2}C_{1a}^{(0)} + d^{(0)}D_{1a}^{(0)}) \qquad (3.1)$$
$$-G_a^{(1)} = \delta^{(0)}(A_{2a}^{(0)} + b^{(0)}B_{2a}^{(0)} + b^{(0)2}C_{2a}^{(0)} + d^{(0)}D_{2a}^{(0)})$$
$$\theta_a^{(1)} = \delta^{(0)}(A_{3a}^{(0)} + b^{(0)}B_{3a}^{(0)} + d^{(0)}C_{3a}^{(0)} + b^{(0)}d^{(0)}D_{3a}^{(0)})$$

Coefficients $A_{ia}{}^{(0)}$, $B_{ia}{}^{(0)}$, $G_{ia}{}^{(0)}$ and $D_{ia}{}^{(0)}$ $(i = 1, 2, 3)$ are defined by the following explicit functions of zero approximation $\{z_m\}$ [1] (the case of $\mu \sim T$ is considered below):

$$A_{1a}^{(0)} = -P_1{}^*\frac{A_0}{A_1}\left[\frac{A_1}{A_2}(z_2 - 1) - \frac{A_{-1}}{A_0}(z_0 - 1) - I_{1.0} + \frac{A_0}{A_1}(z_1 - 1)\right] + \qquad (3.2)$$
$$(P_1{}^* + N_1{}^*)(1 + p_1)\Big(J_{0.0} + \frac{A_0}{A_2}(z_2 - 1) +$$
$$N_1{}^*(1 + p_1)\frac{A_0}{A_2}(z_2 - 1)\,t_0$$

$$B_{1a}^{(0)} = P_2{}^* \left(\frac{A_{-1}}{A_0} I_{0.0} - \frac{A_0}{A_1} I_{1.0} + \left(\frac{A_0}{A_1} - \frac{A_{-1}}{A_0}\right) \frac{A_0}{A_1}(z_1 - 1)\right) + (P_2{}^* + N_3{}^* + \delta_1 N_1{}^*)\left(\frac{A_{-1}}{A_1}(z_1 - 1) - \frac{A_0}{A_2}(z_2 - 1) - J_{0.0} + J_{0.-1}\right) - P_2{}^* \left(\frac{A_{-1}}{A_1}(z_1 - 1) - \frac{A_0}{A_2}(z_2 - 1)\right)$$

$$C_{1a}^{(0)} = (N_2{}^* + \delta_2 N_1{}^*)(J_{0.0} - 2J_{0.-1} + J_{-1.-1})$$

$$D_{1a}^{(0)} = -N_1{}^*(1 - t_0)(1 + p_1)\left(\frac{A_{-1}}{A_1}(z_1 - 1) - \frac{A_0}{A_2}(z_2 - 1)\right)$$

where

$$J_{0.0} = \frac{A_0{}^2}{2A_1{}^2}(1 - z_1{}^2) - \left(\frac{2}{\sqrt{\pi}} + \frac{A_0}{A_1}\right)\frac{A_{-1}}{2A_0}(1 - z_0{}^2) - \frac{1}{\sqrt{\pi}}\frac{A_0}{A_1}(z_1(\sqrt{2}) - 1)$$

$$J_{0.-1} = -\frac{A_{-1}}{2A_0} I_{0.0}, \quad J_{-1.-1} = \frac{\sqrt{\pi}}{4}\frac{A_0}{A_1}(z_1(\sqrt{2}) - 1)$$

We determine in a similar manner the functions

$$A_{2a}^{(0)} = M_1{}^*(1 + p_1)\left(J_{0.0} + 2\frac{A_0}{A_2}(z_2 - 1) - (1 - t_0)\frac{A_0}{A_2}(z_2 - 1)\right) \quad (3.3)$$

$$B_{2a}^{(0)} = p_1{}^*\left[\frac{A_0}{A_2}(z_2 - 1) - 2\frac{A_{-1}}{A_1}(z_0 - 1) + \frac{1}{2}(z_{-1} - 1) + \left(\frac{A_0}{A_1}\right)^2(z_1 - 1) - \frac{A_0}{A_1}(I_{1.0} - I_{1.-1})\right] + (P_1{}^* + M_3{}^* + \delta_1 M_1{}^*) \times \left(\frac{A_{-1}}{A_1}(z_1 - 1) - \frac{A_0}{A_2}(z_2 - 1) - J_{0.0} + J_{0.-1}\right)$$

$$C_{2a}^{(0)} = P_2{}^*\left\{\left[I_{0.-1}\frac{A_{-1}}{A_0} - I_{0.0}\frac{A_{-1}}{A_0} + I_{1.0}\frac{A_0}{A_1} - I_{1.-1}\frac{A_0}{A_1}\right] - \left(\frac{A_0}{A_1} - \frac{A_{-1}}{A_0}\right)\left(\frac{A_0}{A_1}(z_1 - 1) - \frac{A_{-1}}{A_0}(z_0 - 1)\right)\right\} + (P_2{}^* + M_2{}^* + M_1{}^*\delta_2)(J_{0.0} - 2J_{0.-1} + J_{-1.-1})$$

$$D_{2a}^{(0)} = -M_1{}^*(1 - t_0)(1 + p_1)\left(\frac{A_{-1}}{A_1}(z_1 - 1) - \frac{A_0}{A_2}(z_2 - 1)\right)$$

$$A_{3a}^{(0)} = P_1{}^*\left[\frac{A_0}{A_1}\left(\frac{A_1}{A_2}(z_2 - 1) - \frac{A_{-1}}{A_0}(z_0 - 1) - I_{1.0} + \frac{A_0}{A_1}(z_1 - 1)\right) + J_{0.0} + \frac{A_0}{A_2}(z_2 - 1)\right] \quad (3.4)$$

$$B_{3a}^{(0)} = P_2{}^*\left[\frac{A_{-1}}{A_0} I_{0.0} - \frac{A_0}{A_1} I_{1.0} + \left(\frac{A_0}{A_1} - \frac{A_{-1}}{A_0}\right)\frac{A_0}{A_1}(z_1 - 1) + J_{0.0} - J_{0.-1}\right]$$

$$C_{3a}^{(0)} = -P_1{}^*\left[\frac{A_0}{A_2}(z_2 - 1) - 2\frac{A_{-1}}{A_1}(z_0 - 1) + \frac{1}{2}(z_{-1} - 1) + \left(\frac{A_0}{A_1}\right)^2(z_1 - 1) - \frac{A_0}{A_1}(I_{1.0} - I_{1.-1}) + \frac{A_{-1}}{A_1}(z_1 - 1) - \frac{A_0}{A_2}(z_2 - 1) - J_{0.0} + J_{0.-1}\right]$$

$$D_{3a}^{(0)} = -P_2^*\left(I_{0.-1}\frac{A_{-1}}{A_0} - I_{0.0}\frac{A_{-1}}{A_0} + I_{1.0}\frac{A_0}{A_1} - I_{1.-1}\frac{A_0}{A_1} - \right.$$
$$\left.\left(\frac{A_0}{A_1} - \frac{A_{-1}}{A_0}\right)\left(\frac{A_0}{A_1}(z_1 - 1) - \frac{A_{-1}}{A_0}(z_0 - 1)\right) + J_{0.0} - 2J_{0.-1} + J_{-1.-1}\right.$$

From these formulas we can find the expressions for coefficients $A_{ia}^{(0)}$, $B_{ia}^{(0)}$, $C_{ia}^{(0)}$ and $D_{ia}^{(0)}$ $(i = 1, 2, 3)$ for $\zeta \to \infty$. The coefficients of system (2.2) calculated in the first approximation are

$$-A_{1a\infty}^{(0)} = \frac{P_1^*}{4} + N_1^*(1 + p_1)\left(\frac{1}{2\pi} + \frac{t_0}{4}\right) \qquad (3.5)$$

$$-B_{1a\infty}^{(0)} = \left(\frac{1}{4} - \frac{\sqrt{2}}{4}\right)P_2^* + \left(\frac{\sqrt{2}}{4} - \frac{1}{2\pi}\right)(N_3^* + \delta_1 N_1^*)$$

$$C_{1a\infty}^{(0)} = (N_2^* + \delta_2 N_1^*)\left(\frac{\sqrt{2}-1}{2} - \frac{1}{2\pi}\right)$$

$$D_{1a\infty}^{(0)} = \frac{1}{4}N_1^*(1 - t_0)(1 + p_1)$$

$$-A_{2a\infty}^{(0)} = M_1^*(1 + p_1)\left(\frac{1}{2\pi} + \frac{t_0}{4}\right)$$

$$-B_{2a\infty}^{(0)} = \frac{3}{4}(\sqrt{2} - 1)P_1^* + \left(\frac{\sqrt{2}}{4} - \frac{1}{2\pi}\right)(M_3^* + \delta_1 M_1^*)$$

$$-C_{2a\infty}^{(0)} = -P_2^*\left(\frac{\sqrt{2}}{2} + \frac{\pi}{8} - 1\right) - (M_2^* + \delta_1 M_1^*)\left(\frac{\sqrt{2}-1}{2} - \frac{1}{2\pi}\right)$$

$$D_{2a\infty}^{(0)} = {}^1/_4 M_1^*(1 - t_0)(1 + p_1), \qquad A_{3a\infty}^{(0)} = {}^1/_4 P_1^*$$

$$B_{3a\infty}^{(0)} = -\frac{\sqrt{2}-1}{4}P_2^*, \qquad C_{3a\infty}^{(5)} = \frac{3}{4}(\sqrt{2} - 1)P_1^*$$

$$D_{3a\infty}^{(0)} = \left(\frac{2-\sqrt{2}}{2} - \frac{\pi}{8}\right)P_2^*$$

If the geometry of the body and the external flow parameters are known, it is possible to determine coefficients M_i^*, N_i^* and P_k^* $(i = 1, 2, 3; k = 1, 2)$ at any point of the body surface, and then determine $b^{(0)}, c^{(0)}$ and $d^{(0)}$ and the velocity profile in the considered approximation at any point of the boundary layer. The iteration scheme used in actual calculations for the successive determination of $b^{(0)}$, $c^{(0)}$ and $d^{(0)}$ is based on the following relationships:

$$Dd^{(0)} = (0.1592 + 25t_0)\ \ (1 + p_1)N_1^* + 0.1943b^{(0)}(N_3^* + \delta_1 N_1^*) - 0.048(N_2^* + \delta_2 N_1^*)b^{(0)2} \qquad (3.6)$$

$$\delta^{(0)} = (0.25P_1^* - 0.1035b^{(0)}P_2^* + 0.3105d^{(0)}P_1^* - 0.0997 \times b^{(0)}d^{(0)}P_1^*)^{1/2}$$

$$b^{(0)} = (B - (B^2 + 4AC)^{1/2}) / 2A$$

$$A = 0.097\ p_2^* + 0.0478\ (M_2^* + \delta_1 - M_1^*)$$

$$B = 0.3105\ P_1^* + 0.1943\ (M_3^* + \delta_2 M^*_1)$$

$$C = M_1^*(1 + p_1)(0.1592 + 0.25t_0 - 0.25\ (1 - t_0)d^{(0)})$$

$$D = 0.25\ \ (1 + p_1)(1 - t_0)N_1^* + 0.3105P_1^* - 0.0997\ b^{(0)}\ P_2^*$$

Setting $b^{(0)}$ equal zero in the first and second of relationships (3.6) we determine from these $d^{(0)}$ and $\delta^{(0)}$, and then, using the third, we find $b^{(0)}$. This process is repeated until these parameters are determined with the required accuracy. Having solved system (3.6), we determine the components of friction at the wall and, also, the heat flux to the body. In the case of localized self-similarity for any arbitrary geometry of the body and external flow parameters we have

$$l_0 \frac{\partial E}{\partial \lambda}\Big|_{\lambda=0} = \sqrt{\delta^{(0)}}\{0.2336p_1^* + (0.7978 - 0.5642(1-t_0)) \times (1+p_1)N_1^* + b^{(0)}(0.2095N_3^* - 0.1125p_2^* + 0.2095\delta_1 N_1^*) - 0.0711b^{(0)2}(N_2^* + \delta_2 N_1^*) - 0.322d^{(0)}N_1^*(1-t_0)(1+p_1)\} \tag{3.7}$$

$$l_0 \frac{\partial G}{\partial \lambda}\Big|_{\lambda=0} = \sqrt{\delta^{(0)}}\{M_1^*(1+p_1)(0.7978 - 0.5642(1-t_0)) + 0.2095b^{(0)}(P_1^* + M_3^* + \delta_1 M_1^*) - 0.0711b^{(0)2}(P_2^* + M_2^* + \delta_2 M_1^*) - 0.322d^{(0)}M_1^*(1-t_0)(1-p_1)\}$$

$$l_0 \frac{\partial \theta}{\partial \lambda}\Big|_{\lambda=0} = \sigma\sqrt{\delta^{(0)}}\{(P_1^* - R_1^*)(0.2336 + 0.2095d^{(0)}) - b^{(0)}(P_2^* - R_2^*)(0.1125 + 0.0711d^{(0)})\}$$

4. Let us consider the flow in the neighborhood of a three-dimensional stagnation point in a system of coordinates whose origin is located at that point. Velocity components in the neighborhood of the stagnation point can be defined thus:

$$u_e = a\xi, \qquad \omega_e = b\eta$$

We select functions $\alpha(\xi, \eta)$ and $\beta(\xi, \eta)$ as follows:

$$\alpha = \xi, \quad \beta = \eta / \xi, \quad \varphi = \omega_e / \beta u_e = b / a$$

Coefficients M_i^*, N_i^* and P_k^* $(i = 1, 2, 3;\ k = 1, 2)$ are of the form

$$M_1^* = \varphi^2 - \varphi, \quad M_2^* = 1, \quad M_3^* = 2\varphi, \quad N_1^* = 1, \quad N_2^* = 0$$
$$N_3^* = 0, \quad P_1^* = 1 + \varphi, \quad P_2^* = 1$$

The formulas for calculating the coefficient of friction at the wall and of the heat flux to it are of the form $(l_0 = 1)$

$$\frac{\partial E}{\partial \lambda} = [0.4672 + 0.2336\varphi + 0.5642t_0 - 0.1125b^{(0)} - 0.322d^{(0)}(1-t_0)]\, c^{-1} \tag{4.1}$$

$$\frac{\partial G}{\partial \lambda} = [\varphi(\varphi - 1)(0.2336 + 0.5642t_0) + b^{(0)}(0.207 + 0.622\varphi) - 0.1422b^{(0)2} - 0.322d^{(0)}\varphi(\varphi-1)(1-t_0)]\, c^{-1}$$

$$\frac{\partial \theta}{\partial \lambda} = [(1+\varphi)(0.2336 + 0.2073d^{(0)}) - b^{(0)}(0.1125 + 0.0711d^{(0)})]\, c^{-1}$$

where

$$c = (0.25(1+\varphi) - 0.1035b^{(0)} + 0.3105d^{(0)}(1+\varphi) - 0.0997\, b^{(0)}\, d^{(0)})^{1/2}$$

$$b^{(0)} = 3.39[0{,}31 + 0.699\varphi - ((0.31 + 0.699\varphi)^2 + 0.59(\varphi^2 - \varphi)(0.159 + 0.25t_0 - 0.25(1-t_0)d^{(0)}))^{1/2}]$$

$$d^{(0)} = (0.159 + 0.25t_0)/(0.5606 - 0.25t_0 + 0.3105\varphi - 0.0997b^{(0)})$$

Introducing the notation $E + G / \varphi = h_\lambda$ and $E = f_\lambda$, we obtain a system of equations in variables h and f, which is of the same form as given in [3]. Results obtained by using system (4.1) were compared with those presented in [3] obtained by numerical integration of the input system of differential equations. Comparison of these results is presented in Fig. 1 which shows that the first approximation (solid lines) is in good agreement with the results of numerical calculations. The difference method proposed in [3] is not convergent for $\varphi < -0{,}5$.

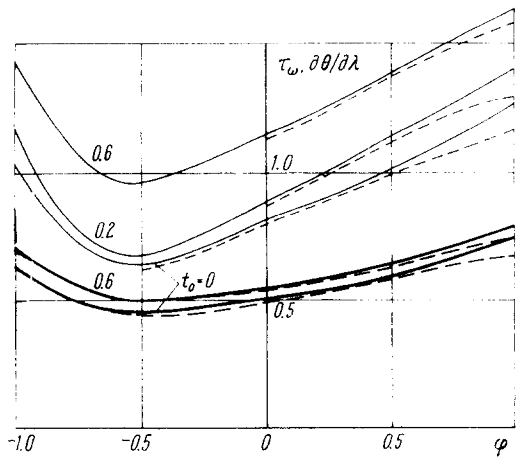

Fig. 1

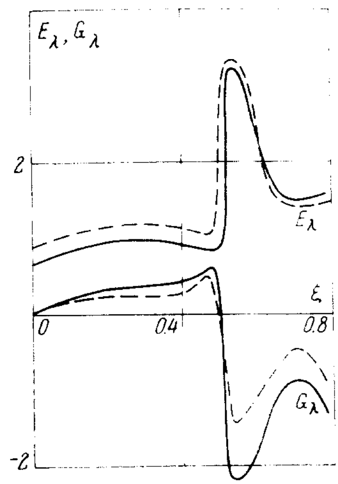

Fig. 2

Thus it is seen that even the first approximation yields good results in the case of a stagnation point of double curvature within the whole range of variation of parameter φ, as well as in a wide range of variation of parameter t_0. The localized self-similar approximation provides not only a qualitative picture of the flow, but also, quantitative data which are in good agreement with those obtained in complicated problems by numerical calculations.

Let us consider the problem of the boundary layer at a segmental body at an angle of attack around which flows a perfect gas. This problem was solved in [2] by the method of finite differences. Let us compare results of numerical calculations with those obtained with the use of formulas (3.7) and (3.6) in a localized self-similar approximation. We assume the external flow to be known, hence coefficients M_i^*, N_i^* and P_k^* can be readily determined $(i = 1, 2, 3; k = 1, 2)$. Using (3.6) we determine functions $b^{(0)}$, $c^{(0)}$ and $d^{(0)}$ and then from (3.7) we obtain the longitudinal and transverse components of friction and of the heat flux to the wall. The problem is solved for the same parameters of the oncoming stream as in [2], i.e. $\alpha = 15°$, $M_\infty = \infty$, $\beta^* = 30°$ and $t_0 = 0{,}5$.

A comparison of results obtained analytically for $\eta = {}^2/_5\,\pi$ in the localized self-similar approximation (solid lines) with those derived by the finite difference method is shown in Fig. 2. It is seen from this comparison that even for an essentially three-dimensional flow the approximate analytical method yields a good correlation.

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THERMAL STRESSES IN THE PLANE PROBLEM OF THE THEORY OF ELASTICITY CAUSED BY PHASE TRANSFORMATIONS

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V. I. GOISA
(Moscow)

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We study the state of stress in an elastic half-plane in the presence of phase transformations caused by temperature variations at the points of the half-plane. Separately we consider the states of stress caused by the lack of homogeneity in the temperature field and the consequent volume changes taking place in the regions of phase transformations.

Under the term "phase transformation" we understand the structural change in the crystal lattice which occurs when the body is heated above a certain critical temperature [1, 2]. Here the purely thermal stresses are accompanied by the stresses associated with the volume change in the region undergoing phase transformations. Similar problems arise during the investigation of the stress states in the case of elastic tension and in the problems on inclusions. Such problems were studied by D. I. Sherman, Iu. A. Amen-Zade, and others. However in all the problems studied the region occupied by an inclusion was always completely contained within some external region.

The present paper deals with the case in which the boundary separating the media has common points with the outer boundary of the region containing the inclusion. The stresses and strains are assumed to satisfy the conditions of the linear theory of elasticity, with the external region and the inclusion possessing the same elastic properties.

Let us assume that a steady-state plane temperature field is applied to the elastic half-plane $y < 0$ and, that the boundary $y = 0$ is free from external forces. Then the stress components satisfy the following boundary conditions:

$$\sigma_y = \tau_{xy} = 0, \qquad y = 0 \tag{1}$$

The temperature field satisfies the boundary value problem for Laplace equation